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The algebraic method for two-dimensional quantum atomic systems

Le Van Hoang and Nguyen Thu Giang

Department of Theoretical Physics, Byelorussian State University, Minsk 220080, Republic of Belarus

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Abstract. We propose an easy and effective variant of the realization of the dynamical symmetry of hydrogen-like atoms in two-dimensional space, based on the relationship between the Schrödinger equation for the isotropic harmonic oscillator in one-dimensional complex space and the Schrödinger equation for the one-electron atom in two-dimensional space, that permits us to use the operator method to solve the Schrödinger equation for the two-dimensional atomic system. For illustration of the algebraic method, we consider the problem of the two-dimensional hydrogenic donor in a magnetic field. By the use of the path integral we also establish the relationship between the two-dimensional Coulomb Green function and the Green function for the isotropic harmonic oscillator in one-dimensional complex space.

1. Introduction

The connection between the problem of hydrogen-like atoms and the problem of the isotropic harmonic oscillator was found long ago (see, for example, Bergmann and Frishman 1965). At the present moment it is well known (Cordani 1989) that the above-mentioned connection was established only for one-, two-, three- and five-dimensional hydrogen-like atoms. Naturally, from them only the connection for three-dimensional hydrogen-like atom is used in a number of atomic problems (Boiteux 1973, Komarov and Romanova 1982, Kibler and Negadi 1983, Iwai 1982, Chetouani and Hammann 1987, etc). Specifically, it was used in Kibler *et al* (1986) and Komarov *et al* (1987) to construct the perturbation method for the solution of the Schrödinger equation for hydrogen-like atoms in an electromagnetic field. Recently the problem of the two-dimensional atomic system has been evoking great interest for a number of authors because of its importance in several practical problems ((Jia-Lin Zhu *et al* 1990, Kramer and Wallis 1988, Greene and Bajaj 1985, etc). One, certainly, can use effectively the well considered connection (for three-dimensional hydrogen-like atom, see Komarov and Romanova 1982) and the operator method (Feranchuk and Komarov 1982) for solving the two-dimensional atomic problems. However, from our point of view, the working out of a new way to use an algebraic method directly for two-dimensional atomic problems is extremely desirable. Therefore in the present paper we develop the connection between the Schrödinger equation for the isotropic harmonic oscillator in one-dimensional complex space and the Schrödinger equation for the one-electron atom in two-dimensional space (section 2), and on the basis of this connection construct the algebraic way to use the oscillator basis functions for calculations of atomic characteristics (section 3). To illustrate the algebraic method, worked

out in section 3, we build the solution of the Schrödinger equation for a two-dimensional hydrogenic donor in a magnetic field (section 4). By the use of the path integral, in section 5 we also establish the relationship between the two-dimensional Coulomb Green function and the Green function of the isotropic harmonic oscillator in one-dimensional complex space that permits one to use the operator form of the latter in concrete calculations.

2. The connection between the Schrödinger equations

Let us consider the Schrödinger equation for the isotropic harmonic oscillator in one-dimensional complex space with the coordinates ξ, ξ^*

$$\mathbf{H}\Psi(\xi) = Z\Psi(\xi) \quad (1)$$

$$\mathbf{H} = -\frac{1}{2} \frac{\partial^2}{\partial \xi \partial \xi^*} + \frac{1}{2} \omega^2 \xi \xi^*. \quad (2)$$

Here the asterisk denotes complex conjugate operation; ω is a real positive number; natural units ($\hbar = c = m = 1$) are used in our paper.

By direct calculations one can show that the operator \mathbf{H} commutes with the operator

$$\mathbf{L} = \frac{1}{2} \left(\xi^* \frac{\partial}{\partial \xi^*} - \xi \frac{\partial}{\partial \xi} \right). \quad (3)$$

The scalar product of wavefunctions in the ξ -space will be defined as:

$$\langle \Psi(\xi) | \psi(\xi) \rangle = \int d\xi' \int d\xi'' \Psi^*(\xi', \xi'') \psi(\xi', \xi'') \quad (4)$$

where $\xi' = \text{Re } \xi$, $\xi'' = \text{Im } \xi$. Thus operators \mathbf{H} and \mathbf{L} are Hermitian with respect to the scalar product (4). Hence equations (1) and (2) can be written as follows

$$\tilde{\mathbf{H}}\tilde{\Psi}(\xi) = -\frac{1}{2}\omega^2\tilde{\Psi}(\xi) \quad (5)$$

$$\tilde{\mathbf{H}} = -\frac{1}{2\xi\xi^*} \frac{\partial^2}{\partial \xi \partial \xi^*} - \frac{Z}{\xi\xi^*}. \quad (6)$$

So $-\frac{1}{2}\omega^2$ can be regarded as an eigenvalue of the operator $\tilde{\mathbf{H}}$. For the operator $\tilde{\mathbf{H}}$ to be Hermitian, the definition of the scalar product for wavefunctions must be changed as follows:

$$\langle \tilde{\Psi}(\xi) | \tilde{\psi}(\xi) \rangle = 4 \int d\xi' \int d\xi'' \tilde{\Psi}^*(\xi', \xi'') \tilde{\psi}(\xi', \xi'') \xi \xi^*. \quad (7)$$

Let us now make a substitution of the variables in equations (5), (6) using the following correlation†

$$\begin{aligned} x_1 &= \frac{1}{2}(\xi^2 + \xi^{*2}) \\ x_2 &= -\frac{i}{2}(\xi^2 - \xi^{*2}) \\ dr &= dx_1 dx_2 = 4d\xi' d\xi'' \xi \xi^*. \end{aligned} \quad (8)$$

† This transformation is a two-dimensional version of what is sometimes called the Kustaanheimo-Stiefel transformation in the three-dimensional case (Cordani 1989, Barut *et al* 1979).

It is easy to see that x_λ ($\lambda = \overline{1, 2}$) are taken as components of the two-dimensional real vector r . After the substitution of variables (8) equations (5), (6) take the form

$$\left\{ -\frac{1}{2}\Delta - \frac{Z}{r} \right\} \tilde{\Psi}(r) = E \tilde{\Psi}(r) \quad (9)$$

and the scalar product of wavefunctions (7) can be rewritten as follows:

$$\langle \tilde{\Psi}(r) | \tilde{\psi}(r) \rangle = \int dr \tilde{\Psi}^*(r) \tilde{\psi}(r). \quad (10)$$

In equation (9) $r = \sqrt{x_\lambda x_\lambda}$, $\Delta = \partial^2 / \partial x_\lambda \partial x_\lambda$, $\lambda = \overline{1, 2}$. From (9), (10) it follows that the function $\tilde{\Psi}(r)$ is a solution of the Schrödinger equation for the two-dimensional hydrogen-like atom with negative energy, if $E = -\frac{1}{2}\omega^2$ and Z is the nuclear charge. In the coordinates r it can be obtained that

$$L = i \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) \quad (11)$$

i.e. the operator L is an angular momentum operator of the two-dimensional hydrogen-like atom.

3. Algebraic method in the solution of the Schrödinger equation

Now we will show that the found connection will enable us to use the oscillator basis function to solve the atomic problems by purely algebraic methods.

Let us define the operators

$$\begin{aligned} a(\omega) &= \sqrt{\frac{\omega}{2}} \left(\xi + \frac{1}{\omega} \frac{\partial}{\partial \xi^*} \right) & a^+(\omega) &= \sqrt{\frac{\omega}{2}} \left(\xi^* - \frac{1}{\omega} \frac{\partial}{\partial \xi} \right) \\ b(\omega) &= \sqrt{\frac{\omega}{2}} \left(\xi^* + \frac{1}{\omega} \frac{\partial}{\partial \xi} \right) & b^+(\omega) &= \sqrt{\frac{\omega}{2}} \left(\xi - \frac{1}{\omega} \frac{\partial}{\partial \xi^*} \right) \end{aligned} \quad (12)$$

where ω is a real positive number. For the case of an isotropic harmonic oscillator like (1), (2) ω is the oscillator frequency. But, as it will be shown in section 4, if we have to deal with an anharmonic oscillator, ω will be represented as a parameter and it may be defined by the special equation (Komarov and Romanova 1982). The operators (12) satisfy the commutation relations

$$[a(\omega), a^+(\omega)] = 1 \quad [b(\omega), b^+(\omega)] = 1 \quad (13)$$

(we have only included non-zero commutators). Using (12), (13) we find that

$$H = \frac{1}{2}\omega(a^+a + b^+b + 1) \quad L = \frac{1}{2}(a^+a - b^+b) \quad (14)$$

and the state vector (in non-normalized form)

$$|\Psi\rangle = (a^+)^{n_1} (b^+)^{n_2} |0(\omega)\rangle \quad (15)$$

(n_1 and n_2 are positive integers) is the solution of the equations

$$H|\Psi\rangle = Z|\Psi\rangle \quad L|\Psi\rangle = m|\Psi\rangle \quad (16)$$

with the eigenvalues

$$Z = \frac{1}{2}\omega(n_1 + n_2 + 1) \quad m = \frac{1}{2}(n_1 - n_2) \quad (17)$$

respectively. The vacuum-state $|0\rangle$ in (15) is determined by the equations

$$a(\omega)|0(\omega)\rangle = 0 \quad b(\omega)|0(\omega)\rangle = 0 \tag{18}$$

and the condition of normalization $\langle 0(\omega)|0(\omega)\rangle = 1$.

From the set of vectors (15) is built the vector $|n, m\rangle$, which belongs to the integral eigenvalue of the angular momentum operator L , as follows:

$$|n, m\rangle = (a^+)^{n+m}(b^+)^{n-m}|0(\omega)\rangle \tag{19}$$

where n is a positive integer, magnetic quantum number m is an integer and satisfies $|m| \leq n$.

From the above-stated formulation of the relationship it can be concluded that the vector (19) is a solution of the Schrödinger equation for the two-dimensional hydrogen-like atom and it can be used as a Coulomb basis function in concrete calculations.

Let us examine the matrix elements of the operators. Matrix elements of the operator A acting in r -space are found, as shown earlier, after transformation of the operator rA into ξ -space, by means of the expression

$$\int dr \Psi_{nm}^*(r) A \Psi_{n'm'}(r) = \langle \Psi(n, m) | rA | \Psi(n', m') \rangle \tag{20}$$

where the state vector $|\Psi(n, m)\rangle$, corresponding to the wavefunction $\Psi_{nm}(r)$ of the discrete spectrum of the hydrogen-like atom in two-dimensional space, is determined by equation (19) and the condition of normalization

$$\langle \Psi(n, m) | r | \Psi(n, m) \rangle = 1. \tag{21}$$

Expression (20) allows us to perform the calculations algebraically without using the explicit form of the wavefunctions in the coordinate representation. We start with a very simple unitary transformation to carry out the transition from one frequency to another:

$$\begin{aligned} |\Psi(\omega')\rangle &= U(\omega', \omega) |\Psi(\omega)\rangle \\ a(\omega') &= U(\omega', \omega) a(\omega) U^{-1}(\omega', \omega) \end{aligned} \tag{22}$$

where

$$U(\omega', \omega) = \exp\{\ln\sqrt{\omega/\omega'}(a(\omega)b(\omega) - a^+(\omega)b^+(\omega))\} \tag{23}$$

or in the normal form

$$\begin{aligned} U(\omega', \omega) &= \exp\left(\frac{\omega' - \omega}{\omega' + \omega} a^+(\omega) b^+(\omega)\right) \\ &\times \exp\left(-[a^+(\omega)a(\omega) + b^+(\omega)b(\omega) + 1] \ln \frac{\omega' + \omega}{2\sqrt{\omega\omega'}}\right) \\ &\times \exp\left(\frac{\omega' - \omega}{\omega' + \omega} a(\omega) b(\omega)\right). \end{aligned} \tag{24}$$

Second, by transforming the operator A into ξ -space and then into the operators a (a^+) and b (b^+), the latter will appear only in the following combinations:

$$\begin{aligned} M(\omega) &= a(\omega)b(\omega) & M^+(\omega) &= a^+(\omega)b^+(\omega) \\ N(\omega) &= a^+(\omega)a(\omega) + b^+(\omega)b(\omega) \\ m_1(\omega) &= \frac{1}{2}(a^2(\omega) + b^2(\omega)) & m_2(\omega) &= -\frac{1}{2}i(a^2(\omega) - b^2(\omega)) \\ m_1^+(\omega) &= \frac{1}{2}(a^{+2}(\omega) + b^{+2}(\omega)) & m_2^+(\omega) &= \frac{1}{2}i(a^{+2}(\omega) - b^{+2}(\omega)) \end{aligned} \tag{25}$$

$$\begin{aligned}n_1(\omega) &= a^+(\omega)b(\omega) + b^+(\omega)a(\omega) \\n_2(\omega) &= i(a^+(\omega)b(\omega) - b^+(\omega)a(\omega)) \\Q(\omega) &= a^+(\omega)a(\omega) - b^+(\omega)b(\omega).\end{aligned}$$

The 10 operators in (25) generate a closed algebra and satisfy the following commutation relations (for brevity we write the operators without ω)

$$\begin{aligned}[M, M^+] &= N+1 & [M, N+1] &= 2M & [N+1, M^+] &= 2M^+ \\[m_i, m_j^+] &= \delta_{ij}(N+1) - i\varepsilon_{ij}Q \\[M, m_i^+] &= n_i & [m_i, M^+] &= n_i \\[m_i, Q] &= -2i\varepsilon_{ij}m_j & [Q, m_i^+] &= 2i\varepsilon_{ij}m_j^+ \\[m_i, N+1] &= 2m_i & [N+1, m_i^+] &= 2m_i^+ \\[M, n_i] &= 2m_i & [n_i, M^+] &= 2m_i^+ \\[m_i, n_j] &= 2\delta_{ij}M & [n_i, m_j^+] &= 2\delta_{ij}M^+ \\[M, Q] &= 0 & [Q, M^+] &= 0 \\[n_i, n_j] &= i2\varepsilon_{ij}Q & [n_i, Q] &= -2\varepsilon_{ij}n_j & [n_i, N] &= 0 \\(i, j &= \overline{1, 2})\end{aligned}\tag{26}$$

where

$$\begin{aligned}\delta_{ij} &= \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \\ \varepsilon_{ij} &= \begin{cases} 1 & i=1 & j=2 \\ 0 & i=j \\ -1 & i=2 & j=1. \end{cases}\end{aligned}\tag{27}$$

By setting up the linear combinations from the operators (25)

$$\begin{aligned}L_{01} &= \frac{1}{2}n_i & L_{03} &= \frac{1}{2}i(M^+ - M) & L_{ij} &= \frac{1}{2}\varepsilon_{ij}Q & L_{04} &= \frac{1}{2}(M^+ + M) \\L_{i3} &= \frac{1}{2}(m_i + m_i^+) & L_{i4} &= \frac{1}{2}i(m_i - m_i^+) & L_{34} &= -\frac{1}{2}(N+1)\end{aligned}\tag{28}$$

and introducing the definition

$$L_{\lambda\mu} = -L_{\mu\lambda} \quad (\lambda, \mu = \overline{0, 4})\tag{29}$$

we obtain, with the aid of (26),

$$[L_{\lambda\mu}, L_{\rho\nu}] = -i(f_{\lambda\rho}L_{\mu\nu} + g_{\mu\nu}L_{\lambda\rho} - g_{\lambda\nu}L_{\mu\rho} - g_{\mu\rho}L_{\lambda\nu})\tag{30}$$

with the metric $(---+)$. Thus we find that the algebra of operators (25) is the Lie algebra of the group $SO(3, 2)$, which is the dynamical symmetry group of the Schrödinger equation for the two-dimensional hydrogen-like atom (see Malkin and Manko 1979). As in Komarov and Romanova (1982) we note that transformation (22) permits us to calculate the matrix elements of the operators, between the state vectors, which are connected with different values of the charge in equation (9).

For further use we write some operators in the creation and annihilation operators form:

$$\begin{aligned}
 x_\lambda &= \frac{1}{2\omega}(m_\lambda + m_\lambda^+ + n_\lambda) & (\lambda = \overline{1, 2}) & & r &= \frac{1}{2\omega}(M + M^+ + N + 1) \\
 r \frac{\partial}{\partial x_\lambda} &= \frac{1}{2}(m_\lambda - m_\lambda^+) & x_\lambda \frac{\partial}{\partial x_\lambda} &= \frac{1}{2}(M - M^+ - 1) \\
 r \frac{\partial^2}{\partial x_\lambda \partial x_\lambda} &= \frac{1}{2}(M + M^+ - N - 1) & i \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) &= \frac{1}{2}Q
 \end{aligned}$$

4. Example of application: two-dimensional hydrogenic donor states in a magnetic field

The Schrödinger equation for a two-dimensional hydrogenic donor in the presence of a magnetic field *B*, which is perpendicular to the two-dimensional plane, can be written as

$$\mathbf{H}\Psi(r) = E\Psi(r) \tag{31}$$

$$\mathbf{H} = -\frac{1}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{1}{2} i\gamma \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) + \frac{1}{8} \gamma^2 (x^2 + y^2) - \frac{Z}{r}. \tag{32}$$

Here we use the atomic units system and

$$E = E/R^* \quad \gamma = \hbar\omega_c/2R^*$$

where $R^* = \mu e^4/2\hbar^2 \epsilon^2$ is the cyclotron frequency; the coordinates are in units of the effective Bohr radius $a^* = \epsilon \hbar^2/e^2 \mu$; μ and ϵ are the electron effective mass and static dielectric constant respectively; *Z* is equal to one.

As shown in sections 2 and 3, we can use the simple algebraic method to solve the equation (31), (32), writing it in the ξ -space by the use of the variables substitution (8). We obtain

$$\left\{ -\frac{1}{2} \frac{\partial^2}{\partial \xi \partial \xi^*} + \frac{1}{8} \gamma^2 (\xi \xi^*)^3 + \frac{1}{4} \gamma \xi \xi^* \left(\xi \frac{\partial}{\partial \xi} - \xi^* \frac{\partial}{\partial \xi^*} \right) - Z - E \xi \xi^* \right\} \Psi(\xi) = 0. \tag{33}$$

Using operators (12), (25) we can rewrite (33) as follows:

$$(\mathbf{A} - E\mathbf{B})|\Psi\rangle = 0$$

$$\mathbf{A} = -\frac{\omega}{2}(M + M^+ - N - 1) + \frac{\gamma^2}{64\omega^3}(M + M^+ + N + 1)^3 + \frac{\gamma}{4\omega}(M + M^+ + N + 1)L - Z \tag{34}$$

$$\mathbf{B} = (M + M^+ + N + 1)/2\omega$$

where ω is an arbitrary real parameter. It is easy to see from (25) and (26) that the operator **L** ($L = \frac{1}{2}Q$) commutes with all operators in the equation (34), so that if we use the suitable basis function as (19), the equation (34) is represented as an equation for a two-dimensional anharmonic oscillator and the well known operator method (Feranchuk and Komarov 1982) can be used for its solution. Thus to build the solution of (34) we use the basis functions (19)

$$|n, m\rangle = \frac{1}{\sqrt{(n+m)!(n-m)!}} (a^+)^{n+m} (b^+)^{n-m} |0(\omega)\rangle. \tag{35}$$

Matrix elements of operators **A** and **B** for basis functions (35) can be obtained algebraically without difficulty using correlation (13)

$$\begin{aligned}
 \mathbf{A}_{jn}^m &= \langle jm | \mathbf{A} | nm \rangle \\
 &= \left(-\frac{\omega}{4} + \frac{m\gamma}{4\omega} + \frac{3\gamma^2}{64\omega^3} (5n^2 + 1 - m^2) \right) \sqrt{n^2 - m^2} \delta_{j,n-1} \\
 &\quad + \left(\frac{\omega}{2} + \frac{m\gamma}{2\omega} + \frac{\gamma^2}{16\omega^3} (5n^2 + 5n + 3 - 3m^2) \right) \left(n + \frac{1}{2} \right) \delta_{jn} - Z\delta_{jn} \\
 &\quad + \left(-\frac{\omega}{4} + \frac{m\gamma}{4\omega} + \frac{3\gamma^2}{64\omega^3} (5n^2 + 10n + 6 - m^2) \right) \sqrt{(n+1)^2 - m^2} \delta_{j,n+1} \\
 &\quad + \frac{\gamma^2}{64\omega^3} (\delta_{j,n-3} \sqrt{(n^2 - m^2)((n-1)^2 - m^2)((n-2)^2 - m^2)} \\
 &\quad + (6n-3) \delta_{j,n-2} \sqrt{(n^2 - m^2)((n-1)^2 - m^2)} \\
 &\quad + (6n+9) \delta_{j,n+2} \sqrt{((n+2)^2 - m^2)((n+1)^2 - m^2)} \\
 &\quad + \delta_{j,n+3} \sqrt{((n+3)^2 - m^2)((n+2)^2 - m^2)((n+1)^2 - m^2})
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{B}_{jn}^m &= \langle jm | \mathbf{B} | nm \rangle \\
 &= (\delta_{j,n-1} \sqrt{(n^2 - m^2)} + (2n+1) \delta_{jn} + \delta_{j,n+1} \sqrt{((n+1)^2 - m^2)}) / 2\omega.
 \end{aligned} \tag{36}$$

The exact solution of (34) can be written as follows:

$$|\Psi_{nm}\rangle = |n, m\rangle + \sum_{\substack{k=|m| \\ k \neq n}}^{\infty} C_k^{nm} |k, m\rangle. \tag{37}$$

Putting (37) into (34) we obtain the equations for E_{nm} and for the coefficients C_k^{nm} :

$$\begin{aligned}
 E_{nm} &= \frac{\mathbf{A}_{nn}^m}{\mathbf{B}_{nn}^m} + \sum_{\substack{k=|m| \\ k \neq n}}^{\infty} C_k^{nm} \frac{\mathbf{A}_{nk}^m - E_{nm} \mathbf{B}_{nk}^m}{\mathbf{B}_{nn}^m} \\
 C_j^{nm} &= - \sum_{\substack{k=|m| \\ k \neq j}}^{\infty} C_k^{nm} \frac{\mathbf{A}_{jk}^m - E_{nm} \mathbf{B}_{jk}^m}{\mathbf{A}_{jj}^m - E_{nm} \mathbf{B}_{jj}^m}.
 \end{aligned} \tag{38}$$

Taking into account the peculiarity of matrix elements \mathbf{A}_{jn} , \mathbf{B}_{jn} , it is easy to see that the sums in (38) are finite by few non-trivial terms and we can solve (38) by the simple iterative method to obtain the solution with any accuracy. The parameter ω can be chosen so that the speed of an iterative convergence is satisfactory (see Chan Za An *et al* 1986). The zero-order of the operator method

$$\begin{aligned}
 |\Psi_{nm}\rangle^{(0)} &= |nm\rangle \\
 E_{nm}^{(0)} &= -\frac{Z\omega}{n + \frac{1}{2}} + \frac{\omega^2}{2} + \frac{1}{2} m\gamma + \frac{\gamma^2}{16\omega^2} (5n^2 + 5n + 3 - 3m^2)
 \end{aligned} \tag{39}$$

as shown in the same problem, like the problem of an anharmonic oscillator (see Feranchuk and Komarov 1984), is very accurate if we choose the parameter ω by the equation

$$\partial E_{nm}^{(0)}(\omega) / \partial \omega = 0 \tag{40}$$

which leads to the equation for ω

$$\omega = \frac{Z}{n + \frac{1}{2}} + \frac{\gamma^2}{8\omega^3} (5n^2 + 5n + 3 - 3m^2). \tag{41}$$

Detailed investigation of the problem of the hydrogenic donor states will be given in another paper. In the present paper we only consider it as an example for illustration of the algebraic method using the connection between the Schrödinger equation for two-dimensional hydrogen-like atoms and the Schrödinger equation for the isotropic harmonic oscillator.

5. The connection between the Green functions

The Green function for equations (1), (2) in the ‘energy’ representation is a solution of the following equation:

$$\left(Z + \frac{1}{2} \frac{\partial^2}{\partial \xi \partial \xi^*} - \frac{1}{2} \omega^2 \xi \xi^* \right) \mathbf{U}(\xi, \eta; Z) = i \delta(\xi' - \eta') \delta(\xi'' - \eta'') \tag{42}$$

where $\delta(x)$ is a Dirac δ -function. One of the ways of constructing the function $\mathbf{U}(\xi, \eta; Z)$ is to represent it as a path integral (see, for example, Slavnov and Fadeev 1978)

$$\mathbf{U}(\xi, \eta; Z) = \int_0^\infty d\vartheta e^{iZ\vartheta} \prod_{\vartheta'} \int D^2 \xi(\vartheta') \exp \left\{ i \int_0^\vartheta d\vartheta' \left(2 \dot{\xi} \dot{\xi}^* - \frac{1}{2} \omega^2 \xi \xi^* \right) \right\} \tag{43}$$

where $\dot{\xi}(\vartheta') = \partial \xi(\vartheta') / \partial \vartheta'$ and $\xi(\vartheta) = \xi$, $\xi(0) = \eta$. Equation (43) is regarded as a limit (when $\varepsilon \rightarrow 0$, $N \rightarrow \infty$, $N\varepsilon \rightarrow \vartheta$) of the following expression:

$$\begin{aligned} \mathbf{U}(\xi, \eta; Z) = & \int_0^\infty d\vartheta \lim_{\substack{\varepsilon \rightarrow 0 \\ N \rightarrow \infty}} \left(\frac{2}{i\pi\varepsilon} \right)^N \prod_{k=1}^{N-1} \int_0^\infty d^2 \xi(k) \\ & \times \exp \left\{ i \sum_{k=1}^N \left(\frac{2}{\varepsilon} (\xi^*(k) - \xi^*(k-1)) (\xi(k) - \xi(k-1)) \right. \right. \\ & \left. \left. + Z\varepsilon - \frac{1}{2} \omega^2 \varepsilon \xi^*(k-1) \xi(k-1) \right) \right\} \end{aligned} \tag{44}$$

where $\xi(N) = \xi$, $\xi(0) = \eta$ and $d^2 \xi(k) = d\xi'(k) d\xi''(k)$ (for brevity, we use the notation $\xi(k) \equiv \xi(k\varepsilon)$). To establish the relationship between the function $\mathbf{U}(\xi, \eta; Z)$ and the two-dimensional Coulomb Green function, we change in (44) the variables, choosing as in (8) the new variables $x_1(\vartheta)$, $x_2(\vartheta)$. By direct calculations we can easily ascertain that

$$\dot{\xi}(\vartheta) \dot{\xi}^*(\vartheta) = \dot{x}_\lambda(\vartheta) \dot{x}_\lambda(\vartheta) / 4r(\vartheta) \tag{45}$$

where $\dot{x}_\lambda(\vartheta) = dx_\lambda(\vartheta) / d\vartheta$ and $r(\vartheta) = \sqrt{x_\lambda(\vartheta) x_\lambda(\vartheta)} = \xi^*(\vartheta) \xi(\vartheta)$. From (8) it follows that in calculating the path integral using (44), the appropriate substitution of variables is

$$\begin{aligned} x_1(k) &= \frac{1}{2} (\xi^2(k) + \xi^{*2}(k)) \\ x_2(k) &= -\frac{1}{2i} (\xi^2(k) - \xi^{*2}(k)) \\ d^2 \xi(k) &= d^2 x(k) / 4r(k). \end{aligned} \tag{46}$$

Therefore it follows

$$\begin{aligned}
 \mathbf{U}(\xi, \eta; Z) = & \int_0^\infty d\vartheta \lim_{\substack{\varepsilon \rightarrow 0 \\ N \rightarrow \infty}} \left(\frac{2}{i\pi\varepsilon} \right) \prod_{k=1}^{N-1} \int \frac{d^2x(k)}{2i\pi\varepsilon r(k)} \\
 & \exp \left\{ i \sum_{k=1}^N \left[\frac{1}{2\varepsilon r(k)} (x_\lambda(k) - x_\lambda(k-1))(x_\lambda(k) - x_\lambda(k-1)) \right. \right. \\
 & \left. \left. + Z\varepsilon - \frac{1}{2}\omega^2 \varepsilon r(k-1) \right] \right\}. \tag{47}
 \end{aligned}$$

The last step is the change of ‘time’ variable: let

$$\varepsilon(k) = \varepsilon r(k) \tag{48}$$

which in the limit $\varepsilon \rightarrow 0$ is equivalent to the introduction of a new time variable

$$t = \int_0^\vartheta d\vartheta' r(\vartheta'). \tag{49}$$

As a result we have

$$\begin{aligned}
 \frac{1}{4}\mathbf{U}(\xi, \eta; Z) & \equiv \mathbf{K}(r, r'; -\frac{1}{2}\omega^2) \\
 & = \int_0^\infty dt \exp\left(-i\frac{\omega^2}{2}t\right) \prod_\tau \int D^2x(\tau) \\
 & \quad \times \exp\left\{ i \int_0^t d\tau \left(\frac{1}{2}x_\lambda(\tau)x_\lambda(\tau) + \frac{Z}{r(\tau)} \right) \right\} \tag{50}
 \end{aligned}$$

where $r(t) = r$ and $r(0) = r'$. Denoting $E = -\frac{1}{2}\omega^2$ we obtain from (50)

$$\left\{ E + \frac{1}{2}\Delta + \frac{Z}{r} \right\} \mathbf{K}(r, r'; E) = i \delta(r - r') \tag{51}$$

(Δ is the Laplace operator), i.e. $\mathbf{K}(r, r'; E)$ is the ‘energy’ representation of the two-dimensional Coulomb Green function. Starting from the well known expression of the Green function of the isotropic harmonic oscillator in one-dimensional complex space

$$\begin{aligned}
 \mathbf{U}(\xi, \eta; Z) = & \frac{\sqrt{2\omega}}{\pi} \int_0^\infty dt \exp\left(-i\frac{2Z}{\omega}t\right) (\sin t)^{-1} \\
 & \times \exp\left\{ \frac{i\omega}{\sin t} ((\xi^*\xi + \eta^*\eta) \cos t - (\xi^*\eta + \eta^*\xi)) \right\} \tag{52}
 \end{aligned}$$

by the changing of the variables as is shown in (8), we obtain the two-dimensional Coulomb Green function

$$\begin{aligned}
 \mathbf{K}(r, r'; E) = & \frac{\sqrt{2\omega}}{4\pi} \int_0^\infty dt \exp\left(i\frac{2Z}{\omega}t\right) (\sin t)^{-1} \\
 & \times \exp\left\{ \frac{i\omega}{\sin t} ((r+r') \cos t - 2\sqrt{rr'} \cos(\varphi - \varphi')) \right\}. \tag{53}
 \end{aligned}$$

The relationship established above allows one to use the operator form of the function $\mathbf{U}(\xi, \eta; Z)$ in concrete calculations and thus to reduce rather complicated calculations

of matrix elements of the Coulomb Green function to purely algebraic procedures of transforming the products of the creation and annihilation operators to a normal form (see Le Van Hoang *et al* 1989).

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